PRINCIPLES OF ANALYSIS
LECTURE 1 - SETS AND FUNCTIONS

PAUL L. BAILEY

1. Sets

Set and *element* are undefined terms, except to the extent that we know the relationship between them is *containment*; elements are contained in sets.

If two symbols a and b represent the same element, we write $a = b$. If the symbols a and b represent different elements, we write $a \neq b$. If an element a is a contained in a set A, this relation is written $a \in A$. If a is not in A, this fact is denoted $a \notin A$. We assume that the statements $a \in A$ and $a = b$ are always either true or false, although we may not know which.

Two sets are considered equal when they contain the same elements:

$$A = B \Leftrightarrow [x \in A \Leftrightarrow x \in B].$$

2. Subsets

Let A and B be sets. We say that B is a *subset* of A and write $A \subset B$ if $x \in B \Rightarrow x \in A$.

It is clear that $A = B$ if and only if $A \subset B$ and $B \subset A$.

A set with no elements is called an *empty set*. Since two sets are equal if and only if they contain the same elements, there is only one empty set, and it is denoted $\varnothing$. The empty set is a subset of any other set.

If X is any set and $p(x)$ is a proposition whose truth or falsehood depends on each element $x \in X$, we may construct a new set consisting of all of the elements of X for which the proposition is true; this set is denoted:

$$\{x \in X \mid p(x)\}.$$

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3. Set Operations

Let X be a set and let $A, B \subset X$.

The *intersection* of A and B is denoted by $A \cap B$ and is defined to be the set containing all of the elements of X that are in both A and B:

$$A \cap B = \{x \in X \mid x \in A \text{ and } x \in B\}.$$

The *union* of A and B is denoted by $A \cup B$ and is defined to be the set containing all of the elements of X that are in either A or B:

$$A \cup B = \{x \in X \mid x \in A \text{ or } x \in B\}.$$

We note here that there is no concept of "multiplicity" of an element in a set; that is, if x is in both A and B, then x occurs only once in $A \cup B$.

The *complement* of A with respect to B is denoted $A \setminus B$ and is defined to be the set containing all of the elements of A which are not in B:

$$A \setminus B = \{x \in X \mid x \in A \text{ and } x \notin B\}.$$

The *symmetric difference* of A and B is denoted $A \triangle B$ and is defined to be the set containing all of the elements X which are in either A or B both not both:

$$A \triangle B = \{x \in X \mid x \in A \cup B \text{ and } x \notin A \cap B\}.$$

Proposition 1. *Let X be a set and let $A, B, C \subset X$. Then*

- $A = A \cup A = A \cap A$;
- $\varnothing \cap A = \varnothing$;
- $\varnothing \cup A = A$;
- $A \subset B \Leftrightarrow A \cap B = A$;
- $A \subset B \Leftrightarrow A \cup B = B$;
- $A \cap B = B \cap A$;
- $A \cup B = B \cup A$;
- $(A \cap B) \cap C = A \cap (B \cap C)$;
- $(A \cup B) \cup C = A \cup (B \cup C)$;
- $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$;
- $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$;
- $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$;
- $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$;
- $A \subset B \Rightarrow A \cup (B \setminus A) = B$;
- $A \subset B \Rightarrow A \cap (B \setminus A) = \varnothing$;
- $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap B \cap C)$;
- $(A \setminus B) \setminus C = A \setminus (B \cup C)$;
- $A \triangle B = (A \cup B) \setminus (A \cap B)$;
- $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

4. Cartesian Product of Two Sets

If a and b are elements, we can construct a new element

$$(a, b) = \{\{a\}, \{a, b\}\},$$

called an *ordered pair*. Ordered pairs obey the "defining property":

$$(a, b) = (c, d) \Leftrightarrow a = c \text{ and } b = d.$$

If (a, b) is an ordered pair, then a is called the *first coordinate* and b is called the *second coordinate*.

Let A and B be sets. The *cartesian product* of A and B is denoted $A \times B$ and is defined to be the set of ordered pairs whose first coordinate is in A and whose second coordinate is in B:

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Proposition 2. *Let X be a set and let $A, B, C \subset X$. Then*

- $(A \cup B) \times C = (A \times C) \cup (B \times C)$;
- $(A \cap B) \times C = (A \times C) \cap (B \times C)$;
- $A \times (B \cup C) = (A \times B) \cup (A \times C)$;
- $A \times (B \cap C) = (A \times B) \cap (A \times C)$;
- $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$.

5. Functions

Let A and B be sets. A *function* from A to B is a subset $f \subset A \times B$ such that

$$\forall a \in A \exists! b \in B \vdash (a, b) \in f.$$

If f is such a subset of $A \times B$, we indicate this fact by writing $f : A \to B$. If $a \in A$, the unique element of b such that $(a, b) \in f$ is denoted $f(a)$. Functions obey the "defining property":

- for every $a \in A$ there exists $b \in B$ such that $f(a) = b$;
- if $f(a) = b$ and $f(a) = c$, then $b = c$.

Let $f : A \to B$ be a function. The *domain* of f is A, and the *codomain* of f is B.

If $C \subset A$, the *image* of C is $f(C) = \{b \in B \mid f(c) = b \text{ for some } c \in C\}$. The *range* of a function is the image of its domain.

If $D \subset B$, the *preimage* of D is $f^{-1}(D) = \{a \in A \mid f(a) \in D\}$.

We say that f is *injective* (or *one to one*) if for every $a_1, a_2 \in A$ we have $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$.

We say that f is *surjective* (or *onto*) for every $b \in B$ there exists $a \in A$ such that $f(a) = b$. A function is surjective if and only if its range is equal to its codomain.

We say that f is *bijective* if it is both injective and surjective.

If A is a set, define the *identity function* on A to be the function $\mathrm{id}_A : A \to A$ given by $\mathrm{id}_A(a) = a$ for all $a \in A$. This function is bijective.

If $f : A \to B$ and $g : B \to C$, define the *composition* of f and g to be the function $g \circ f : A \to B$ given by $g \circ f(a) = g(f(a))$.

We say that f is *invertible* if there exists a function $f^{-1} : B \to A$, called the *inverse* of f, such that $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$.

Proposition 3. *A function is invertible if and only if it is bijective.*

If f is injective, we define the *inverse* of f to be a function $f^{-1} : f(A) \to A$ by $f^{-1}(y) = x$, where $f(x) = y$. Since an invertible function is bijective, it is injective, and this definition of inverse agrees with our previous one in this case.

If $f : A \to B$ is a function and $C \subset A$, we define a function $f \restriction_C : C \to B$, called the *restriction* of f to C, by $f \restriction_C (c) = f(c)$. If f is injective, then so is $f \restriction_C$.

6. Cardinality

We say that two sets have the same *cardinality* if and only if there is a bijective function between them.

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the set of natural numbers and for $n \in \mathbb{N}$ let $H_n = \{m \in \mathbb{N} \mid m < n\}$. A set X is called *finite* if there exists a surjective function from X to H_n for some $n \in \mathbb{N}$. If there exists a bijective function $X \to H_n$, we say that the cardinality of X is n, and write $|X| = N$.

A set X is called *infinite* if there exists an injective function $\mathbb{N} \to X$.

Proposition 4. *A set is infinite if and only if it is not finite.*

Proposition 5. *Let A be a finite set and let $f : A \to A$ be a function. Then f is injective if and only if f is surjective.*

Proposition 6. *Let A and B be finite sets. Then $|A \times B| = |A| \cdot |B|$.*

Department of Mathematics & CSCI, Southern Arkansas University
E-mail address: plbailey@saumag.edu